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Finite Horizon Static Output H_∞ Control with Automatic Beam Guidance Application

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The problem of designing static output-feedback controllers for finite time horizon is considered. The design problem is solved in two settings: one is the pure H_∞ setting where the disturbance attenuation problem is solved, and the other is a mixed H_2/H_∞ setting where an upper bound on the L_2 norm of the error is minimized as well. In the pure H_∞ setting, a single Riccati-like equation should be solved, where the required projection matrix is obtained up to a free parameter. In the other setting, a two-point boundary-value problem has to be solved. The theory of the mixed H_2/H_∞ setting is illustrated by a design of an automatic lateral beam guidance controller.

I. Introduction

THE problem of designing feedback controllers that minimize the energy gain of linear systems is a natural extension[1,2] of the frequency-domain approach to the H_∞ design for the linear time-invariant case.[3] The order of the H_∞ controller is equal to the sum of the orders of the controlled plant and the dynamic weightings. Sometimes, in practical cases, simpler controllers are preferred, at the cost of some compromise on the performance. For example, in designing flight control systems, engineers prefer the simple and physically sound controllers that are recommended as cooked structures (see, e.g., Refs. 4 and 5). In these simple structures of controllers, only gains (and sometimes integrators for the sake of steady-state performance) are included and, thus, the closed-loop poles are obtained by the migration of the open-loop poles that have a clear physical meaning (i.e., phugoid and short-period modes in the longitudinal dynamics, Dutch-roll and spiral modes in lateral dynamics, etc.). In Ref. 6, an H_∞/minimum-entropy static output-feedback control theory has been developed for the infinite horizon (time-invariant case) situation. The theory there has been applied to a simple flight control design.

A large amount of research has been devoted to infinite horizon, H_∞ static-output feedback and mixed H_2/H_∞ control of time-invariant systems. We refer the reader to Refs. 6 and 7 and the references therein for a review of these works. Unfortunately, not all of the control problems are solvable by static-output feedback. This issue is also addressed in Ref. 6.

In the present paper, the static output-feedback problem for finite time horizon, time-varying case is considered in an H_∞ setting. First, the analog of the standard H_∞ problem is treated, where an energy-gain bound can be ensured, via static output-feedback, by solving a control type, modified, Riccati-like matrix differential equation with an end condition. Then, in analogy to the infinite horizon case,[6] an upper bound on an H_2 property of the closed-loop system (which is closely related to the system entropy in the infinite time horizon case[8]) is minimized, while ensuring the energy-gain bound. The solution is obtained in terms of two strongly coupled equations. The first is a control type, modified, Riccati-like matrix differential equation, with an end condition and the second is a filtering type, matrix Lyapunov differential equation with an initial condition.

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The mixed H_2/H_∞ static output-feedback approach is applied to design an automatic lateral beam guidance loop, which is a part of an automatic landing system.

II. Problem Formulation

Consider the following system:

$$\dot{x} = Ax + B_1 w + B_2 u \tag{1a}$$

$$y = C_2 x \tag{1b}$$

$$z = C_1 x + D_{12} u \tag{1c}$$

where we assume that C_2 is of full rank and

$$D_{12}^T [D_{12} \quad C_1] = [R \quad 0] \tag{2}$$

with

$$R > 0 \tag{3}$$

The problem is to find a static output-feedback controller

$$u = K(t) \cdot y \tag{4}$$

so that for all $w \in L_2[0, T]$, and all finite $x(0)$, the following inequality holds:

$$J = \|z\|_2^2 - \|w\|_2^2 - x^T(0) R_0^{-1} x(0) \leq 0, \qquad R_0 > 0 \tag{5}$$

The performance index J is the standard objective function of H_∞ control. It describes the difference between the weighted energy of the regulation error and the control effort and the energy of the exogenous inputs. Condition (5) implies dissipation of the input energy through the system with a worst-case gain that is less than one. Our problem is a finite horizon static output-feedback version of the dynamic (full-order) output-feedback problem.[6] Whereas in Ref. 6 the condition of Eq. (5) also possesses a frequency-domain interpretation,[9] in our time-varying case this interpretation does not hold. Here, for a zero x_0, a nonpositive J implies that the energy of the controlled output z is always less than or equal to the energy of the disturbance signal w.

As in Ref. 6, more than one controller may exist that satisfies Eq. (5). One way to choose between different controllers is to select the one that minimizes the entropy of the closed-loop transfer function matrix. In the infinite time horizon, dynamic (full-order) output-feedback case, it is well known[8] that the central controller minimizes this entropy. The counterpart of Ref. 8 for the static output-feedback case is given in Ref. 6. It would be nice to solve the analog of the problem of Ref. 6 also for the finite-horizon case. However, the generalization of the entropy for finite time horizon systems, in terms of time-domain properties, is not apparent. Instead, we choose, as in Ref. 10, to distinguish between the different controllers that lead to Eq. (5), by the upper bound they achieve on a certain H_2 property of the closed loop. Namely, if w is a sum of a standard (zero mean, unit intensity) white noise process, and the system initial state is a zero mean random vector that is independent of w, we shall minimize some index of performance J' that satisfies

$$E\left(\int_0^T z^T z \, \mathrm{d}t\right) \leq J' \tag{6}$$

In the sequel, we refer to the problem of finding a controller that leads to Eq. (5) as the static output-feedback finite horizon H_∞ control, and to the problem of minimizing J' (yet to be specified) as the static output-feedback, finite horizon, mixed H_2/H_∞ control.

III. Static Output-Feedback, Finite Horizon H_∞ Control

In this section we derive conditions for the existence of a solution to the static output-feedback control problem and derive the corresponding controller if these conditions are met. Two theorems are given; the proofs of these theorems are given in the Appendix.

We consider any $Y = Y^T \geq 0$ for which $C_2 Y C_2^T > 0$, and we denote

$$\nu \triangleq Y C_2^T \left(C_2 Y C_2^T\right)^{-1} C_2 = C_2^\dagger C_2 \tag{7}$$

where $C_2^\dagger$ is a right inverse of C_2 (namely, $C_2 C_2^\dagger = I$), and we find that

$$\nu^2 = \nu \tag{8}$$

We also denote

$$\nu_\perp \triangleq I - \nu \tag{9}$$

Theorem 1. The requirement of Eq. (5) is satisfied for all w in $L_2[0, T]$ and all finite $x(0)$ if there exists $X(t)$, $t \in [0, T]$, which is a solution of

$$-\dot{X} = A^T X + XA + XB_1 B_1^T X + C_1^T C_1 - XB_2 R^{-1} B_2^T X + \nu_\perp^T X B_2 R^{-1} B_2^T \nu_\perp \tag{10a}$$

$$X(T) = 0 \tag{10b}$$

$$X(0) < R_0^{-1} \tag{10c}$$

One controller that satisfies Eq. (5) is given by

$$K = -R^{-1} B_2^T X Y C_2^T \left(C_2 Y C_2^T\right)^{-1} \tag{11}$$

for Y that satisfies

$$C_2 Y C_2^T > 0 \tag{12}$$

For a given ν, which is parametrized by Y via Eq. (7), with $C_2 Y C_2^T > 0$, Theorem 1 provides a sufficient condition for the existence of Eq. (5). This condition can be easily verified by solving Eqs. (10a–10c) and by testing whether $X(t)$ is bounded in $[0, T]$; if such $X(t)$ exists, it is positive semidefinite.[10] A necessary and sufficient condition is given in the following theorem.

Theorem 2. The requirement of Eq. (5) is satisfied for all $w \in L_2[0, T]$, and all finite $x(0)$, iff there exist $X(t)$ and $L(t)$, $t \in [0, T]$ that satisfy

$$-\dot{X} = A^T X + XA + XB_1 B_1^T X + C_1^T C_1 - XB_2 R^{-1} B_2^T X + L^T R L \tag{13a}$$

$$X(T) = 0 \tag{13b}$$

$$X(0) < R_0^{-1} \tag{13c}$$

$$R^{-1} B_2^T X \nu_\perp = L \nu_\perp \tag{13d}$$

In such a case, any controller that is given by

$$K = \left(L - R^{-1} B_2^T X\right) Y C_2^T \left(C_2 Y C_2^T\right)^{-1} \tag{14}$$

leads to Eq. (5).

That Theorem 1 provides only a sufficient condition implies that there may be a special choice for Y and, correspondingly, a special projection matrix ν that provides better closed-loop properties (e.g., a smaller H_∞ norm, that is, achieving a nonpositive J, where B_1 is replaced by $\gamma^{-1} B_1$ for γ smaller than one). On the other hand, Theorem 2 provides a necessary and sufficient condition. As such it leads to an H_∞ norm that may be lower than the one achieved by Theorem 1. Unfortunately, finding the optimal $L(t)$ in Theorem 2 is not straightforward. An alternative approach is to apply the result of Theorem 1 for γ that is larger than its minimum achievable value γ_0 and to find ν that minimizes the H_2 norm of the closed-loop subject to the H_∞ bound of γ.

In the next section, one such special structure of ν is exploited via a dual interpretation of w. Namely, the extra degree of freedom in the selection of Y is used to minimize an upper bound on an H_2 property of the closed loop.

IV. Static Output-Feedback, Finite Horizon Mixed H_2/H_∞ Control

In this section we derive a solution to the mixed H_2/H_∞ static output-feedback control problem for finite horizon, time-varying systems. The solution is given in a theorem, which is proved in the Appendix.

To motivate our problem formulation, we begin by stating a result about the H_2 norm of an H_∞-norm bounded linear system in a finite time horizon setup. This result is also proved in the Appendix.

Lemma 1. Consider the system of Lemma A.2 (see the Appendix) with $J \le 0, \forall \boldsymbol{w} \in L_2[0, T]$. If $\boldsymbol{w}$ is a standard white noise process, rather than a process in $L_2[0, T]$, and $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is a random initial state, which is independent of w with zero mean and $E\{\boldsymbol{x}_0\boldsymbol{x}_0^T\} = Y_0$, then,

$$E\left(\int_0^T \|z\|^2\,\mathrm{d}t\right) \le \mathrm{tr}\left(\int_0^T \tilde{B}^T X \tilde{B}\,\mathrm{d}t\right) + \mathrm{tr}\{X(0)Y_0\}$$

where X is the solution of Eq. (A2) subject to Eq. (A3).

Motivated by Lemma 1 we define the following cost function:

$$J' = \mathrm{tr}\{X(0)Y_0\} + \mathrm{tr}\left(\int_0^T B_1^T(t)X(t)B_1(t)\,\mathrm{d}t\right) \qquad (15)$$

We note that J' will serve as the required bound in Eq. (6).

We obtain the main result of this section.

Theorem 3. The requirement of Eq. (5) is satisfied for all $\boldsymbol{w} \in L_2[0, T]$ if there exist $X(t)$ and $Y(t)$ that satisfy Eqs. (10a–10c), (12), and

$$\dot{Y} = \left(A + B_2KC_2 + B_1B_1^TX\right)Y + Y(A + B_2KC_2 + B_1B_1X)^T + B_1B_1^T \qquad (16a)$$

$$Y(0) = Y_0 \qquad (16b)$$

where the static output-feedback gain K is given by Eq. (11). The gain matrix K minimizes J' of Eq. (15) only if X and Y satisfy these conditions.

We note here that if the system matrices are all constant and X and Y converge, for T that goes to infinity, to their steady-state values, then J' of Eq. (15) will be unbounded. If, however, we scale J' by T^{-1}, we obtain in the limit, where T goes to infinity, the entropy of the closed loop in the sense of Refs. 6 and 8.

We also note that the solution to the mixed H_2/H_∞ problem is associated with a two-point boundary-value problem (TPBVP). Solving such a problem is not an easy task but can be tackled using numerical tools such as BOUNDSCO.[11] In Sec. V, we solve a numerical example using an iterative algorithm for solving the TPBVP. Although in our example the iteration process converges, we do not have a convergence proof for the general case. We have applied the following iterative algorithm.

Step 1) Solve Eq. (10a) with the end condition (10b) and with $v(t) = C_2^\dagger C_2$, where $C_2^\dagger$ is the Penrose–Moore pseudoinverse of C_2. The resulting $X(t)$ is stored at each integration step.

Step 2) Solve Eq. (16a) with the initial condition of Eq. (16b) and with $X(t)$ of step 1. Store $Y(t)$.

Step 3) Solve Eqs. (10a) and (10b) with $Y(t)$ of step 2.

Step 4) Repeat steps 2 and 3 until $X(t)$ and $Y(t)$ converge.

V. Example: Automatic Lateral Beam Guidance

A. Problem Description

Our example is taken from Ref. 5 and deals with the design of an automatic lateral beam guidance system for a bank-to-turn aircraft, which can perform coordinated turns. In fact, this lateral beam guidance system can provide the lateral corrections to the aircraft in an automatic landing system.

The control loop we design employs time-varying gains on the slope λ, where λ is measured by a localizer placed in the station. The geometry of the guidance problem, from a top view, is shown in Fig. 1. The distance between the aircraft and the station is $\Gamma(t)$. When $\Gamma(t) = \Gamma_0$, the wheels already touch the ground, and we do not have to worry about the loop performance.

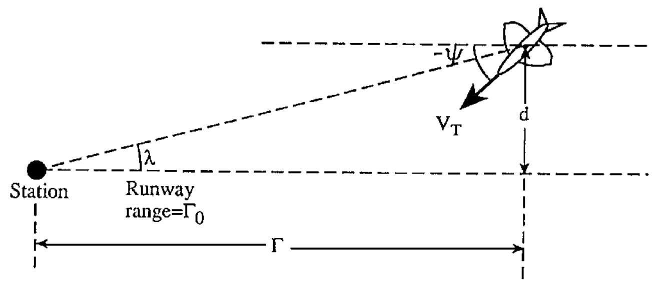

Fig. 1 Geometry of the guidance problem.

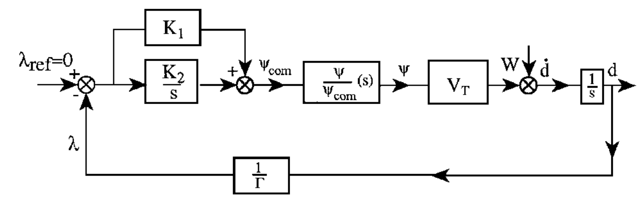

Fig. 2 System dynamics.

B. Controller Structure

Our task is to design a coupler transference that is constructed to have the structure of

$$\Psi_{\mathrm{com}} = -\left[K_1(t)\lambda + K_2\int_0^t \lambda\,\mathrm{d}\upsilon\right]$$

It will process the slope λ, measured by the localizer device, which is positioned at the station, and it will provide an azimuth angle command Ψ_{com} to regulate the distance of the aircraft off course d, in the presence of crosswinds W, and initial conditions $d(0)$. In Ref. 5, the loop gains were taken to be

$$K_1 = 10, \qquad K_2 = 1$$

These gains were set in Ref. 5 to obtain a reasonable dynamic response to initial conditions and crosswinds with a zero steady-state error for d. In this example we fix K_2 to be $K_2 = 1$, as in Ref. 5, and we design a time-varying gain $K_1(t)$ so as to improve the transient response of the closed-loop system.

C. Plant Realization

The dynamics of the control problem is shown in Fig. 2. As in Ref. 5, we consider the case of $V_T = 440$ ft/s, for which the transfer function Ψ/Ψ_{com} (s) (that includes an outer $\Delta\Psi \to \Phi_c$ loop and an inner Φ/Φ_c bank control loop) is given in Ref. 5,

$$\frac{\Psi}{\Psi_{\mathrm{com}}}(s) = \frac{40.4}{(s^2 + 1.4s + 0.64)(s^2 + 11s + 60.8)}$$

Denoting the state-space realization of the latter transfer function by $C_\Psi(sI - A_\Psi)^{-1}B_\Psi$, we obtain the following state-space description of the open-loop dynamics of the system of Fig. 2, augmented by the constant gain controller K_2s^{-1}:

$$\dot{\boldsymbol{x}} = \tilde{A}\boldsymbol{x} + B_1w + B_2u, \qquad y = C_2\boldsymbol{x}$$

where

$$\boldsymbol{x} = [\boldsymbol{x}_1, x_2, x_3]^T, \qquad w = W, \qquad u = -K_1(t)\lambda = -K_1(t)C_2x$$

$$C_2 = [0 \quad \Gamma^{-1} \quad 0]$$

$$A = \begin{bmatrix} A_\Psi & 0 & 0 \\ V_tC_{1\Psi} & 0 & 0 \\ 0 & \Gamma^{-1} & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B_\Psi \\ 0 \\ 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and where

$$\tilde{A} = A - B_2[0 \quad K_2]\tilde{C}_2, \qquad \tilde{C}_2 = \begin{bmatrix} 0 & \Gamma^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\boldsymbol{x}_1$ is the state vector of the dynamics of $(\Psi/\ \Psi_{\mathrm{com}})(s)$,

$$x_2 = d$$

and

$$x_3 = \int_0^t \lambda(\upsilon)\,\mathrm{d}\upsilon$$

To minimize the effect of the initial condition on x_3, which corresponds to requiring a zero steady-state error in λ and also in d, at $\Gamma \to 0$ we choose the minimized signal z to be

$$z = C_1\boldsymbol{x} + D_{12}u$$

where

$$C_1 = \begin{bmatrix} 0 & \Gamma^{-1} & 0.1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 0 \\ \rho \end{bmatrix}, \qquad \rho^2 = 0.005$$

$$R_0 \to 0, \qquad Y_0 = \mathrm{diag}\{0_{4\times 4}, 100, 0\}$$

Thus, we look only for the range-dependent $K_1(t)$ to minimize the H_2 norm of the closed loop, while ensuring the H_∞-norm bound.

D. Gain Computations

The control gain $K_1(t)$ has been obtained by solving Eq. (10a) with the end condition (10b) backward in time and Eq. (16a) with the initial condition (16b), where B_1 is replaced by $\gamma^{-1}B_1$. The disturbance attenuation constant γ was chosen to be $\gamma = 0.1$.

In our example we had to repeat step 4 of the algorithm of Sec. IV three times until convergence has been obtained.

The resulting $K_1(t)$ is shown in Fig. 3.

Remark. One shortcoming of our method is the amount of memory it requires to run the iterative algorithm. If memory is a problem, more effective algorithms, in the style of Appendix B of Ref. 11, should be considered. The latter applies a multiple shooting technique. Shooting refers to the way by which TPBVPs are solved by choosing an initial value and by using, for example, Newton iterations to find the initial value that also leads to the required end value. In the multiple shooting technique, the time interval is divided to a set of subintervals where in each a shooting is applied so that continuity is maintained when the resulting solutions are pieced together.

E. Simulations

The performance of the resulting automatic lateral beam guidance loop was compared with two other designs: 1) the loop of Ref. 5 (i.e., with $K_1 = 10$ and $K_2 = 1$) and 2) a full-state feedback H_∞ controller, which is obtained by taking $C_2 = I$ and, therefore solving Eqs. (19a) and (19b) with $\nu = I$. For this design, the same values for C_1 and D_{12} were taken.

All three loops were tested with an initial condition effect of $d(0) = 500$ ft. That is, the aircraft has to null this initial error considerably before the wheels touch the runway (at $\Gamma = \Gamma_0 = 8205$ ft) and with a reasonable transient behavior. The performance of these three different designs is compared in Figs. 4–6. The path that the aircraft makes is shown in Fig. 4. It is seen that the textbook classical solution of Ref. 5 is slower than the mixed $H_2/\ H_\infty$ static output-feedback design that is somewhat more oscillatory. The faster response of the static output-feedback design is achieved with initially larger and faster azimuth angle command Ψ_{com} (see Fig. 5). The latter is also seen in Fig. 6, where the slope error λ is depicted. These initially large commands are due to K_1, which attains initial values larger than those in Ref. 5 but decreases significantly toward landing.

The reason for the initial relatively large values of K_1 is the small R_0 that we chose. This choice forces the worst-case initial condition $\boldsymbol{x}_0$ to be of small norm, which allows an increase in K_1 without significantly increasing the weighted control energy in J. If a large initial K_1 is undesirable, a larger R_0 should be taken. We note that $K_1(t)$ can be decreased throughout the loop's operation if a larger ρ is chosen.

As could be expected, the full state-feedback design is considerably better than the design of Ref. 5 and the static output-feedback

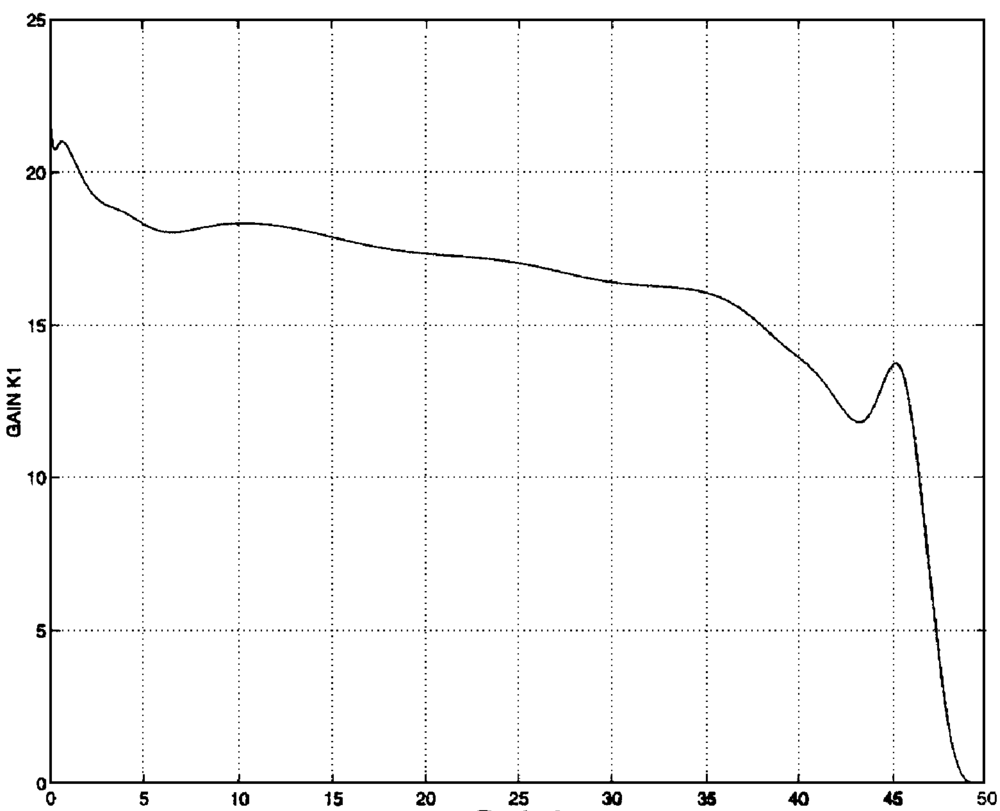

Fig. 3 Controller gain $K_1(t)$ for the mixed H_2/H_∞ design.

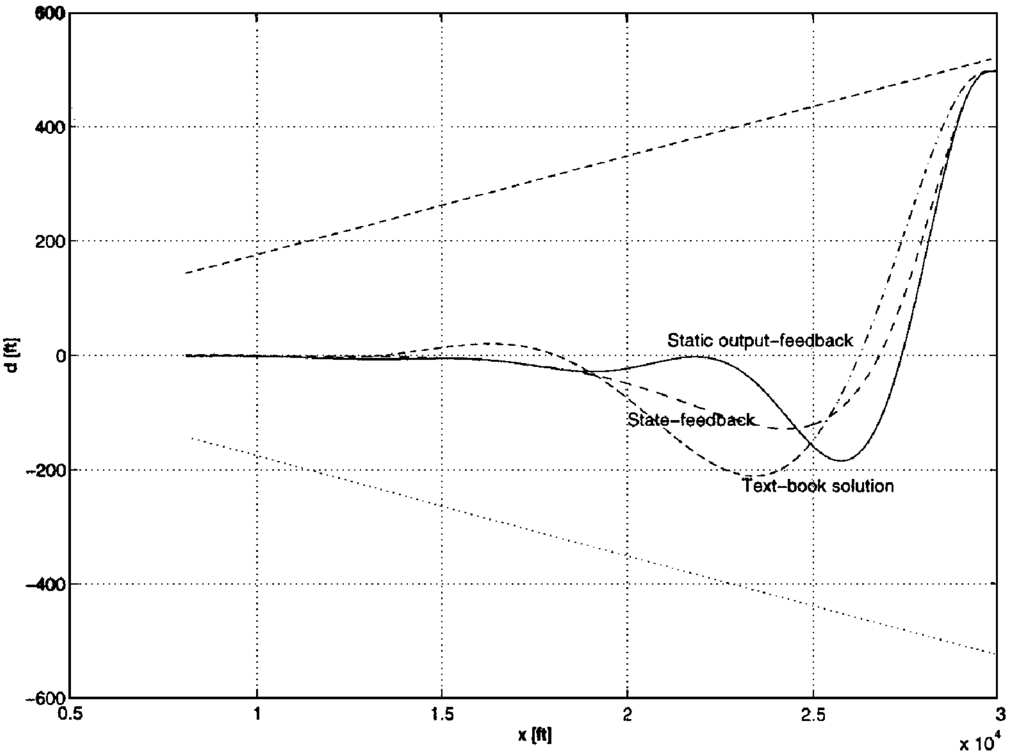

Fig. 4 Aircraft path: comparison of three different controllers. Effect of 500-ft initial cross-track distance.

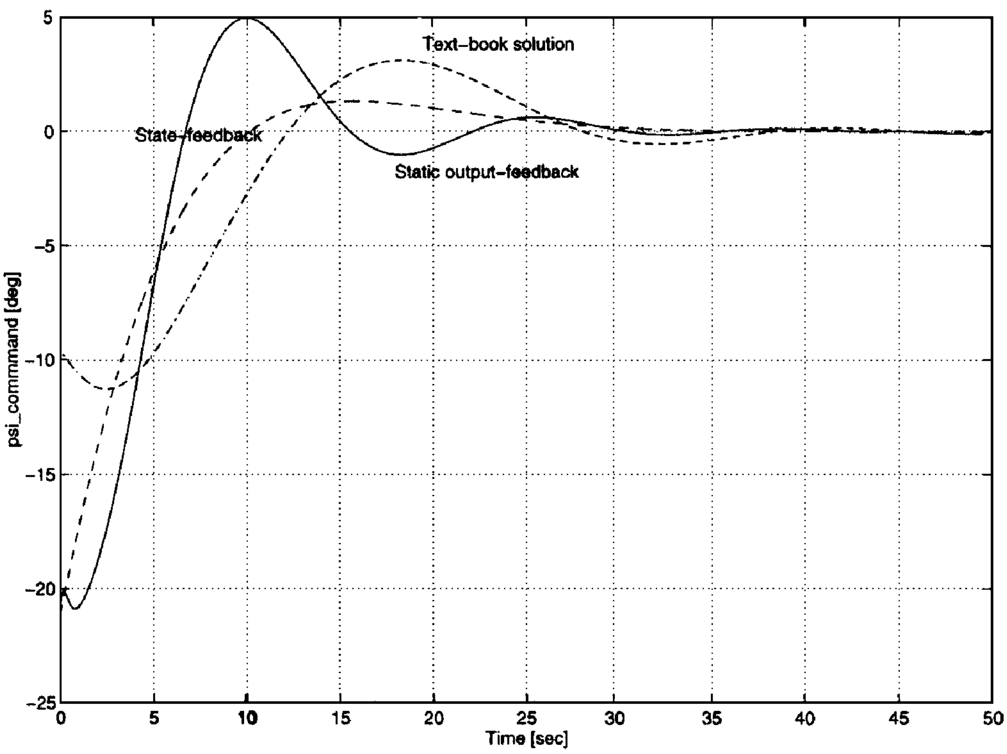

Fig. 5 Azimuth command: comparison of three different controllers. Effect of 500-ft initial cross-track distance.

design. It possesses a fast nonoscillatory response (see Fig. 4), with faster but smaller Ψ_{com} requirements (see Fig. 5). These results may encourage incorporation of additional measurements to improve the static output design. One additional measurement could be the measurement of Ψ, which is always available onboard. Other possibilities are the measurements of the roll angle of the aircraft and the yaw rate. We did not apply the latter measurements in our design example because it is intended to show how a classically structured beam guidance loop can be improved by using time-varying gains only.

Another way to compare the designs quantitatively is to compute

$$J^*(t) = \int_0^t \left\{ \|C_1(\upsilon)x(\upsilon)\|^2 + \|D_{12}(\upsilon)u(\upsilon)\|^2 \right\} \mathrm{d}t$$

where we note that $J^*(T) = \|z\|_2^2$. The comparison of $J^*(t)$ for all three designs is shown in Fig. 7. As could be expected, the lowest J^* is the one of the full state-feedback design followed by the static output-feedback design and the textbook solution of Ref. 5. In this sense, the advantage of the full state-feedback controller over our

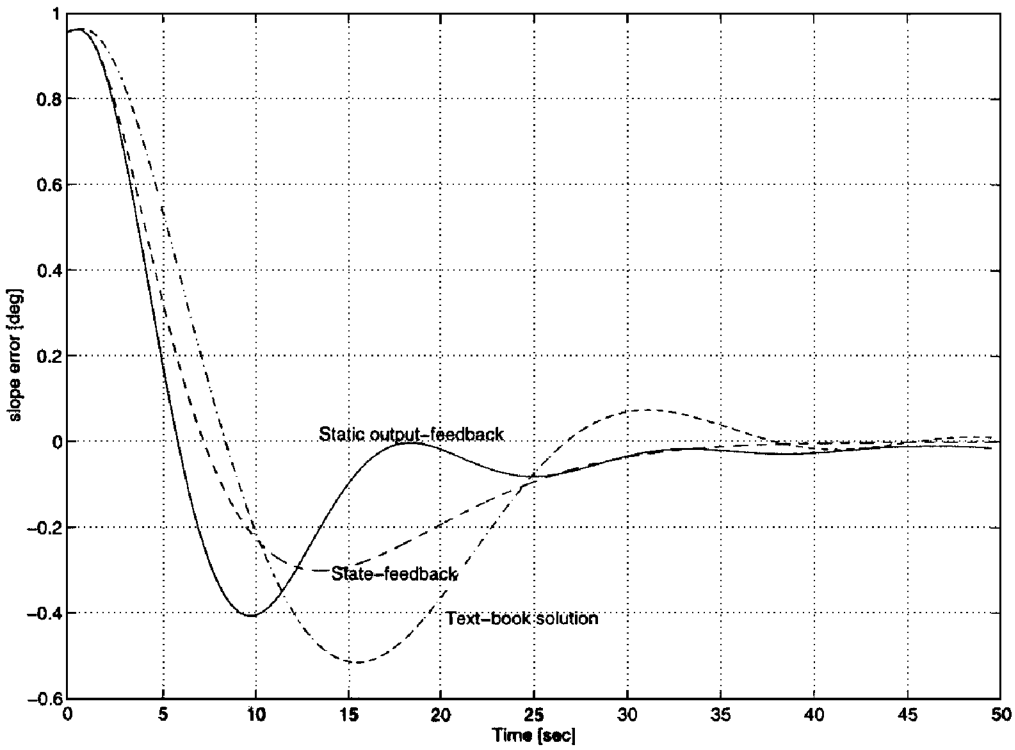

Fig. 6 Slope error: comparison of three different controllers. Effect of 500-ft initial cross-track distance.

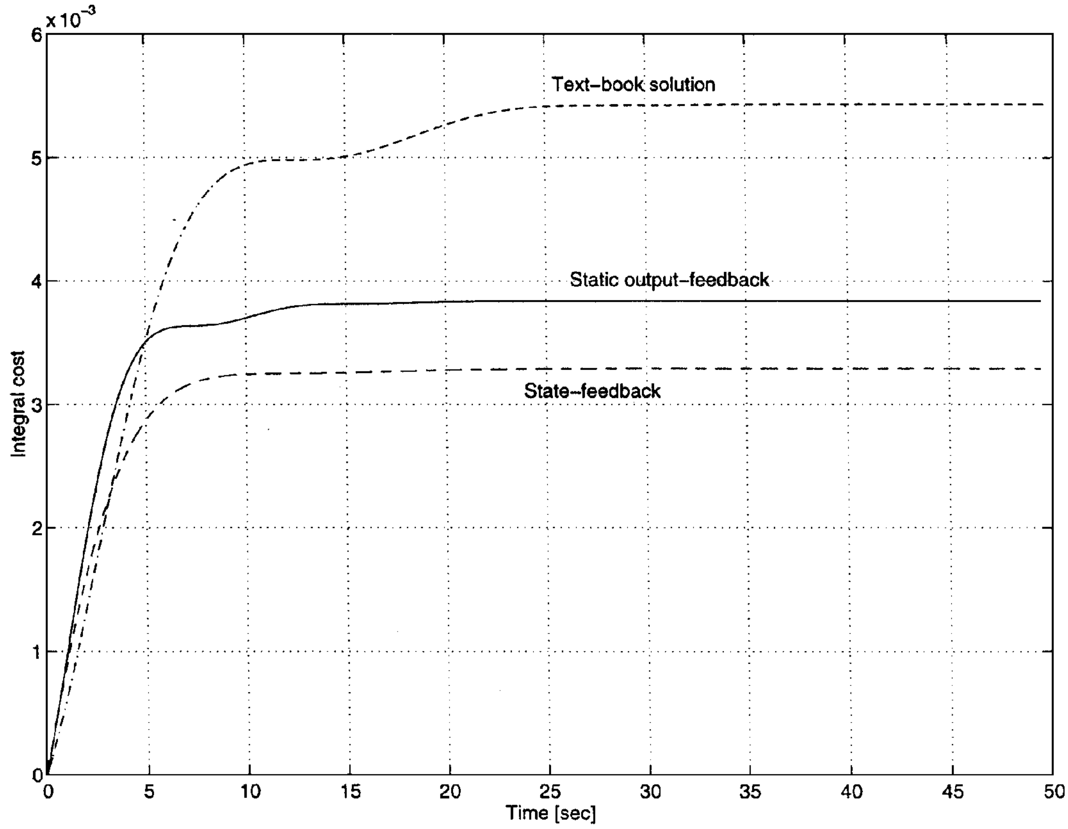

Fig. 7 Integral cost of the three controllers.

static output-feedback controller is relatively small compared to the difference between the state-feedback design and the textbook solution.

VI. Conclusions

The problem of designing static output-feedback controllers that satisfy an energy-gain bound for finite horizon situations has been solved. Our results are based on solving a control type, Riccati-like matrix differential equation, with an end condition, coupled with a filtering type, Lyapunov-like equation with an initial condition.

Unlike the standard H_∞ control problem, no apparent way to decouple these equations emerges. That is why we have used an iterative procedure to obtain the solutions to these equations. We have applied this procedure to design an automatic lateral beam guidance loop. The results are most encouraging, and they motivate the application of the mixed H_2/H_∞ static-feedback solution to many other practical finite time-varying problems.

Appendix: Proofs

Proof of Theorem 1. The proof is based on the following two lemmas.

Lemma A1 (Ref. 12).

1) There exists $K(t)$ in Eq. (4) that satisfies Eq. (5), iff there exists K_c in

$$u = K_c x$$

that satisfies Eq. (5) and $K_c Y_2 = 0$, where the columns of Y_2 constitute an orthonormal basis for the null-space of C_2 (namely, $C_2 Y_2 = 0$, where $Y_2^T Y_2 = I$).

2) If such K_c exists, then for any $Y = Y^T > 0$, the static output-feedback controller

$$K = K_c Y C_2^T \left(C_2 Y C_2^T\right)^{-1} \tag{A1}$$

leads to Eq. (5).

Lemma A2 (Ref. 1). Consider the system

$$\dot{x} = \tilde{A}x + \tilde{B}w, \qquad z = \tilde{C}x$$

and

$$J = \|z\|_2^2 - \|w\|_2^2 - x^T(0)R_0^{-1}x(0)$$

Then, $J \leq 0$ for all $w \in L_2$ and for all finite $x(0)$ iff there exists $X(t)$ so that

$$-\dot{X} = \tilde{A}^T X + X\tilde{A} + X\tilde{B}\tilde{B}^T X + \tilde{C}^T\tilde{C} \tag{A2}$$

where

$$X(T) = 0 \tag{A3a}$$

$$X(0) < R_0^{-1} \tag{A3b}$$

□

Using Eq. (A1) and the definition of Eq. (7), we express KC_2 by

$$KC_2 = K_c v \tag{A4}$$

Substituting Eq. (A4) in Eqs. (1) and (4), we have

$$u = K_c v x \tag{A5}$$

Denoting

$$\tilde{A} = A + B_2 K_c v \tag{A6a}$$

$$\tilde{B} = B_1 \tag{A6b}$$

$$\tilde{C} = C_1 + D_{12} K_c v \tag{A6c}$$

we obtain from Lemma A2 that Eq. (5) is satisfied, iff there exists $X(t)$, $t \in [0, T]$, so that

$$-\dot{X} = A^T X + XA + XB_1B_1^T X + C_1^T C_1 - XB_2R^{-1}B_2^T X + v_\perp^T XB_2R^{-1}B_2^T X v_\perp + S \tag{A7}$$

and

$$X(T) = 0 \tag{A8a}$$

$$X(0) < R_0^{-1} \tag{A8b}$$

where

$$S = \left[K_c^T + XB_2R^{-1}\right]R\left[K_c + R^{-1}B_2^T X\right] + v_\perp^T K_c^T R\left(K_c + R^{-1}B_2^T X\right) + \left(K_c^T + XB_2R^{-1}\right)RK_c v_\perp \tag{A9}$$

The proof readily follows from the fact that by Eqs. (11) and (A4) the corresponding K_c is $-R^{-1}B_2^T X$, which nullifies S.

Proof of Theorem 2. Substituting $u = K_c x$ in Eqs. (1a) and (1c), and using Lemma A2, we find that Eq. (5) is satisfied iff there exists $X(t)$, $t \in [0, T]$, so that

$$-\dot{X} = A^T X + XA + XB_1B_1^T X + C_1^T C_1 - XB_2R^{-1}B_2^T X + \left[K_c^T + XB_2R^{-1}\right]R\left[K_c + R^{-1}B_2^T X\right] \tag{A10a}$$

$$X(T) = 0 \tag{A10b}$$

$$X(0) < R_0^{-1} \tag{A10c}$$

Now, if there exists K_c that satisfies Eq. (5), we take

$$K_c + R^{-1}B_2^T X = L \tag{A11}$$

and obtain Eqs. (13a) and (13b) from Eqs. (A10a) and (A10b). Equation (13d) follows from the fact that by Lemma A1, $K_c v_\perp = 0$. The sufficiency part readily follows by substituting Eq. (A11) in Eq. (13).

Proof of Lemma 1. Considering, first, the contribution of w, it is well known (see, e.g., Ref. 9) that

$$E\left(\int_0^T \|z\|^2 \,dt\right) = \mathrm{tr}\left(\int_0^T \tilde{B}^T Q\tilde{B}\,dt\right)$$

where Q is the observability Grammian that satisfies

$$-\dot{Q} = \tilde{A}^T Q + Q\tilde{A} + \tilde{C}^T\tilde{C}, \qquad Q(T) = 0$$

Subtracting the last equation from Eq. (A2) we get

$$-\frac{d}{dt}(X - Q) = \tilde{A}^T(X - Q) + (X - Q)\tilde{A} + X\tilde{B}\tilde{B}^T X$$

$$X(T) - Q(T) = 0$$

The latter equation is associated with the following autonomous system:

$$\dot{x} = \tilde{A}x, \qquad z = \tilde{B}^T X x, \qquad x(0) = x_0$$

Differentiating $x^T(X - Q)x$ along the trajectory of this system and defining $\Phi(t, \upsilon)$ to be its transition matrix, we obtain that

$$X(t) - Q(t) = \int_t^T \Phi^T(\upsilon, t)X\tilde{B}\tilde{B}^T X\Phi(\upsilon, t)\,d\upsilon \geq 0$$

Evaluating, next, the contribution of a nonzero x_0, we readily find that

$$E\left(\int_0^T \|z\|^2\,dt\right) \leq \mathrm{tr}\{X(0)Y_0\}$$

The lemma follows by considering the combined effect of x_0 and w and using the fact that the two are independent.

Proof of Theorem 3. Substituting in Lemma A2, the matrices

$$\tilde{A} = A + B_2KC_2, \qquad \tilde{B} = B_1, \qquad \tilde{C} = C_1 + D_{12}KC_2$$

and adding

$$J' = \mathrm{tr}\left(\int_0^T \tilde{B}^T X\tilde{B}\,dt\right) + \mathrm{tr}\{X(0)Y_0\}$$

to the dynamic constraint of Eq. (A2) via the Lagrange multiplier Y, we obtain the following augmented cost function:

$$J_c = \int_0^T \mathrm{tr}\Big\{B_1^T XB_1 + \big[\dot{X} + (A + B_2KC_2)^T X + X(A + B_2KC_2) + XB_1B_1^T X + C_1^T C_1 + C_2^T K^T RKC_2\big]Y\Big\}\,dt + \mathrm{tr}\{X(0)Y_0\}$$

$$= \mathrm{tr}\int_0^T \Big\{B_1^T X B_1 + \big[(A+B_2KC_2)^T X + X(A+B_2KC_2) + XB_1B_1^T X + C_1^T C_1 + C_2^T K^T R K C_2\big]Y\Big\}\,\mathrm{d}t - \int_0^T \mathrm{tr}\{X\dot{Y}\}\,\mathrm{d}t + \mathrm{tr}\{X(T)Y(T) - X(0)Y(0)\} + \mathrm{tr}\{X(0)Y_0\}$$

Differentiating J_c with respect to X, we obtain that

$$\frac{\partial J_c}{\partial X} = \mathrm{tr}\int_0^T \Big\{B_1B_1^T + Y(A+B_2KC_2)^T + (A+B_2KC_2)Y + B_1B_1^T XY + YXB_1B_1^T - \dot{Y}\Big\}\,\mathrm{d}t$$

For stationarity, we thus require Y to satisfy Eq. (16a). Because

$$\frac{\partial J_c}{\partial X_0} = -Y(0) + Y_0$$

we also require for stationarity that $Y(0) = Y_0$.

The stationarity with respect to K implies that

$$\mathrm{tr}\int_0^T \Big\{2C_2YXB_2 + 2C_2YC_2^T K^T R\Big\}\,\mathrm{d}t = 0$$

from which K of Eq. (11) follows. The proof is completed by noting that the condition of the theorem satisfies the conditions of Theorem 1 and, therefore, Eq. (5) is satisfied.

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